International Journal of Theoretical Physics, Vol. 45, No. 4, April 2006 (© 2006)
DOI: 10.1007/s10773-006-9068-5

Nonpropagating Solitary Waves in (2 + 1)-Dimensional Generalized Dispersive Long Wave Systems

Dai Chao-qing,[1] Zong Feng-de,[1] and Zhang Jie-fang[1,2]

Received May 6, 2005; accepted January 26, 2006
Published Online April 12, 2006

An extended mapping approach is used to obtain a new type of variable separation excitation, with three arbitrary functions, of the (2 + 1)-dimensional generalized dispersive long wave equation (DLWE). By selecting appropriate functions, the richness of nonpropagating solitons, such as nonpropagating dromion, nonpropagating ring, nonpropagating lump, and nonpropagating foldon, etc., is displayed for the (2 + 1)-dimensional generalized dispersive long wave equation (DLWE) in this paper. Meanwhile, we conclude that the solution v_1 and v_2 are essentially equivalent to the "universal" formula.

KEY WORDS: extended mapping approach; generalized DLW systems; nonpropagating soliton.

PACS numbers: 02.30.Jr, 02.30.Ik, 05.45.Yv

1. INTRODUCTION

Recently, in the study of higher dimensional nonlinear systems, noticeable progress has been made on the propagating solitons (Tang *et al.*, 2002; Tang and Lou, 2003a; Zheng and Sheng, 2003; Zhang, 2001, 2002; Zhang *et al.*, 2001; Zheng and Chen, 2004). However, the real natural world is very colorful. In some cases, it is not enough to describe the natural phenomena merely by the propagating solitons. For instance, Wu *et al.* (1984) reported about the nonpropagating hydrodynamical breather solitons in their experiment. Denardo *et al.* (1990) have also observed a kink in the phase of surface wave oscillations on a shallow liquid in a parametrically driven rectangular channel. The forced standing or nonpropagating solitary wave phenomena, such as breather and kinks as mentioned above, can be explained by the cubic noninear Schrodinger equation (NLS; Drazin and

[1] Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, China.
[2] To whom correspondence should be addressed at Institute of Nonlinear Physics, Zhejiang Normal University, Jinhua 321004, China; e-mail: jf_zhang@zjnu.cn.

Johnson, 1989) which were formulated by Larraza and Putterman (1984) and Miles (1984).

Though there are many investigations on nonpropagating solitary wave, both in theoretical and experimental aspects (Chen *et al.* 1995; Wang *et al.*, 1986; Wu, 1985; Yan and Yang, 1989). to the best of our knowledge, less work has been done to investigate the nonpropagating solitons in higher dimensional nonlinear physical systems. In this paper, we investigate the existence of these phenomena for the celebrated (2 + 1)-dimensional generalized dispersive long wave equation (DLWE)

$$
\begin{aligned}
u_{ty} + (v_x + uu_y)_x &= 0, \\
v_t + (uv + u + u_{xy})_x &= 0,
\end{aligned}
\tag{1}
$$

which was introduced by Ablowitz and Clarkson (1991). The (1 + 1)-dimensional DLWE ($y = x$ of (1)) is called the classical Boussinesq equation. There exist a large number of papers to discuss the possible applications and exact solutions of the (1 + 1)-dimensional DLWE (Chen and Lou, 2003; Musette and Conte, 1994; Musette *et al.*, 1995). Various interesting properties of the (2 + 1)-dimensional DLWE have been studied by many authors (Lou, 1993, 1994, 1995; Paquin and Winternitz, 1990; Tang and Lou, 2003b). For example, Paquin and Winternitz (1990) showed that the symmetry algebra of Eq. (1) is infinite-dimensional and KacMoody-Virasoro structure. Lou (1995) outlined nine types of two-dimensional similarity reductions. Tang and Lou (2003b) have already proved that the (2 + 1)-dimensional DLWE also possesses a quite "universal" formula

$$
U \equiv \frac{-2(a_0a_3 - a_1a_2)p_xq_y}{(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{2}
$$

which is valid for the suitable physical fields or potentials for a large type of (2 + 1)-dimensional physically interesting nonlinear models, such as the Davey–Stewartsen (DS) equation, the dispersive long wave (DLW) equation, the Broer–Kaup–Kupershimidt (BKK) system, the Nizhnik–Novikov–Veselov (NVV) equation, the nonintegrable (2 + 1)-dimensional Korteweg–de Vries(KdV) equation, the general (N+M)-component AKNS system, and the (2 + 1)-dimensional sine-Gordon equation, and so on (Hong *et al.*, 2003; Ruan and Chen, 2001, 2003; Tang *et al.*, 2002; Tang and Lou, 2003a,b; Zhang *et al.*, 2001; Zhang, 2001, 2002; Zheng and Sheng, 2003; Zheng and Chen, 2004) which is now called multi-linear variable separation approach (MLVSA). In expression (2), $p \equiv p(x, t)$ is an arbitrary function of $\{x, t\}$, $q \equiv q(y, t)$ may be either an arbitrary function of $\{y, t\}$ or an arbitrary solution of a Riccati equation, while a_0, a_1, a_2 and a_3 are taken as constants. And we find that the solution v_1 and v_2 in our following discussion are essentially equivalent to the "universal" formula (2). However, as far as we can see, its nonpropagating solitons were not reported in previous literature.

Many powerful methods for searching for the solitary wave solutions to nonlinear evolution equations (NEE) have been proposed. Among them, the extended mapping approach (Peng, 2003) is one of the most effective straightforward methods to construct soliton excitations of NEEs. The basic idea of this approach is as follows. Consider a given nonlinear partial differential equation (NPDE) with independent variables, $x = (t, x_1, x_2, \ldots, x_n)$, and dependent variable u,

$$N(t, x_i, u, u_t, u_x, u_{x_i}, u_{x_i x_j}, \ldots) = 0. \tag{3}$$

We seek for its solution of the form

$$u = \sum_{i=0}^{n} a_i(x)\phi^i(q(x)), \tag{4}$$

with ϕ satisfying the equation

$$\phi' = \phi^2 + p\phi, \tag{5}$$

where p is a constant and the prime denotes the derivative with respect to q. To determined u explicitly, we take following three steps:

Step 1: Determine n by balancing the highest nonlinear terms and the highest-order partial differential terms in the given NPDE (3).

Step 2: Substituting (4) with (5) into (3) and eliminating all the coefficients of the powers of ϕ to obtain a set of partial differential equations for $a_i (i = 0, 1, \ldots, n)$ and q, from which a_i and q are determined.

Step 3: As (5) possesses the general solutions

$$\phi = -\frac{p}{2}\left\{1 + \tan h\left[\frac{p}{2}(q - q_0)\right]\right\}. \tag{6}$$

where $-p < f < 0$ as $p > 0$ and $0 < f < p$ as $p < 0$, and

$$\phi = -\frac{p}{2}\left\{1 + \cot h\left[\frac{p}{2}(q - q_0)\right]\right\}. \tag{7}$$

where $f < p$ or $f > 0$ as $p > 0$ and $f > -p$ or $f < 0$ as $p < 0$, and q_0 is an integral constant. For convenience, we take $q_0 = 0$ in the following discussion. Substituting a_i, q and (6) or (7) into (4), one can obtain possible solution of (3).

2. NEW EXACT SOLUTIONS OF THE (2 + 1)-DIMENSIONAL GENERALIZED DISPERSIVE LONG WAVE EQUATION

In this section, we apply the method to Eq. (1). By the balancing procedure, ansatz (4) becomes

$$\begin{aligned} u &= f + g\phi(q), \\ v &= F + G\phi(q) + H\phi^2(q), \end{aligned} \tag{8}$$

where f, g, F, G, H and q are functions of $X = (x, y, t)$ to be determined. Substituting (8) with (5) into (1) and eliminating all the coefficients of the powers of ϕ, we get

$$6gq_yqx^2 + 3gq_xH = 0, \tag{9}$$

$$6Hqx^2 + 3g^2q_yq_xc = 0, \tag{10}$$

$$\begin{aligned} &4H_xq_x + 2Gq_x{}^2 + 2\,Hq_{xx} + 10\,Hpqx^2 + 2gq_yqt + 2\,fgq_yq_x \\ &\quad + g(g_xq_y + 3gpq_yq_x + g_yq_x + gq_{xy}) \\ &\quad + (g_x + gpq_x)gq_y + gq_x(g_y + gpq_y) = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &g_xq_yq_x + 2gq_xq_{xy} + 2gq_yq_{xx} + g(12pq_yq_x + 2q_{xy})q_x + 2Hq_t \\ &\quad + 2fHq_x + g(Gq_x + H_x + 2Hpq_x) \\ &\quad + 2g_yq_x{}^2 + (g_x + gpq_y)H + gq_xG = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &g_tq_y + 4H_xpq_x + f_xgq_y + (g_x + gpq_x)(g_y + gpq_y) + gq_xf_y + 2G_xq_x \\ &\quad + 3gpq_yq_t + 3Gpq_x{}^2 + 2Hpq_{xx} + 4Hp^2q_x{}^2 + gq_{ty} + H_{xx} \\ &\quad + f(g_xq_y + 3gpq_yq_x + g_yq_x + gq_{xy}) + g(g_{xy} + g_xpq_y \\ &\quad + g_ypq_x + gpq_{xy} + gp^2q_yq_x) + g_yq_t + Gq_{xx} = 0, \end{aligned} \tag{13}$$

$$\begin{aligned} &g_{xx}q_y + Gq_t + 6\,g_xpq_yq_x + gg_x + g(7\,p^2\,q_yq_x + 3\,pq_{xy})q_x + 3\,gpq_yq_{xx} \\ &\quad + 3\,gpq_xq_{xy} + f(Gq_x + H_x + 2Hpq_x) + g(G_x + Gpq_x) + 3g_ypq_x{}^2 \\ &\quad + 2g_xq_{xy} + H_t + 2\,Hpq_t + f_xH + (g_x + gpq_x)G \\ &\quad + gq_xF + gq_{xxy} + 2g_{xy}q_x + g_yq_{xx} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} &g_ypq_t + g_{ty} + g_tpq_y + Gp^2q_x{}^2 + G_{xx} + gpq_{ty} + gp^2q_yq_t + f_x(g_y + gpq_y) \\ &\quad + f(g_{x,y} + g_xpq_y + g_ypq_x + gpq_{xy} + gp^2g_yq_x) + gf_{xy} \\ &\quad + (g_x + gpq_x)f_y + 2G_xpq_x + Gpq_{xx} = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &G_t + Gpq_t + g_x + gpq_x + g_ypq_{xx} + g_{xxy} + g_{xxy}pq_y + g_yp^2qx^2 \\ &\quad + f(G_x + Gpq_x) + gF_x + 2g_xp^2q_yq_x + gp^2q_yq_{xx} \\ &\quad + g(p^3q_yq_x + p^2q_{xy})q_x + f_xG + (g_x + gpq_x)F + gp^2q_xq_{xy} \\ &\quad + 2\,g_{xy}pq_x + 2\,g_xpq_{xy} + gpq_{xxy} = 0, \end{aligned} \tag{16}$$

$$f_{ty} + ff_{xy} + f_xf_y + F_{xx} = 0, \tag{17}$$

$$F_t + f_x F + f_x + f_{xxy} + f F_x = 0. \tag{18}$$

Based on the Eqs. (9) and (10), we have

$$H = -2q_x q_y, \quad g = \pm 2q_x. \tag{19}$$

substituting (19) into (11) and (12), yield

$$G = -2q_{xy} - 2pq_x q_y, \quad f = \frac{\pm q_{xx} \pm qx^2 p - q_t}{q_x}. \tag{20}$$

Reducing Eqs. (13)–(18) by using (19) and (20), we obtain

$$F = -\frac{\mp q_x q_{ty} + qx^2 p q_{xy} + q_x q_{xxy} - q_{xx} q_{xy} \pm q_{xy} q_t + qx^2}{qx^2}, \tag{21}$$

with

$$\begin{aligned} &{q_x}^3 q_{tty} + 2\,qx^3 q_{txxy} - 9\,{q_{xx}}^3 q_{xy} + qx^3 q_{xxxxy} - qx^5 p^2 q_{xxy} - qx^4 p^2 q_{xx} q_{xy} \\ &\quad + q_x q_{xxy} qt^2 - qx^2 q_{xxxx} q_{x,y} - qx^2 q_{xy} q_{tt} + 4\,{q_{xx}}^2 q_x q_{ty} - 12{q_{xx}}^2 q_{xy} q_t \\ &\quad - 3q_{xx} q_{xy} qt^2 - 4\,q_{xx} qx^2 q_{xxxy} - 2\,q_t {q_x}^2 q_{xxxy} - 4qx^2 q_{xxx} q_{xxy} \\ &\quad - 2qx^2 q_{xxx} q_{ty} - 2\,qx^2 q_{xy} q_{txx} - 4\,q_{tx} qx^2 q_{xxy} + 9\,{q_{xx}}^2 q_x q_{xxy} - 2\,qx^2 q_{txy} q_t \\ &\quad - 2q_{tx} qx^2 q_{ty} - 4\,qx^2 q_{txy} q_{xx} + 8\,q_x q_{xy} q_{tx} q_{xx} + 4\,q_x q_{xy} q_{tx} q_t \\ &\quad + 8q_{xx} q_x q_{xxx} q_{xy} + 8q_{xx} q_x q_{xxxy} q_t + 2\,q_t q_{xx} q_x q_{ty} + 4\,q_t q_x q_{xxx} q_{xy} = 0. \end{aligned} \tag{22}$$

It is obvious that to obtain the general solution of the Eq. (22) is difficult. Fortunately, it is straightforward to find that the basic simple ansatz can be taken as

$$q = \chi(x) + \varphi(y) + \tau(t), \tag{23}$$

where $\chi \equiv \chi(x)$, $\varphi \equiv \varphi(y)$, $\tau \equiv \tau(t)$ are three arbitrary variable separation functions of x, y and t, respectively.

Substituting (19)–(21) and (23) into (8) with the solutions of the Eq. (5), we can derive the solution of the Eq. (1), namely

$$u_1 = \frac{\pm\chi_{xx} - \tau_t}{\chi_x} \pm p\chi_x \tan h\left[\frac{p}{2}(\chi + \varphi + \tau)\right], \tag{24}$$

$$v_1 = -1 + \frac{p^2}{2}\chi_x\varphi_y - \frac{p^2}{2}\chi_x\varphi_y \tan h^2\left[\frac{p}{2}(\chi + \varphi + \tau)\right], \tag{25}$$

$$u_2 = \frac{\pm\chi_{xx} - \tau_t}{\chi_x} \pm p\chi_x \cos h\left[\frac{p}{2}(\chi + \varphi + \tau)\right], \tag{26}$$

$$v_2 = -1 + \frac{p^2}{2}\chi_x\varphi_y - \frac{p^2}{2}\chi_x\varphi_y \cos h^2\left[\frac{p}{2}(\chi + \varphi + \tau)\right], \tag{27}$$

where $\chi(x)$, $\varphi(y)$, $\tau(t)$ are three arbitrary variable separation functions.

In fact, the solutions (25) and (27) in our paper is essentially equivalent to the quite "universal" formula (2) by selecting the function $p(x,t)$, $q(y,t)$ and constants a_0, a_1, a_2, a_3 properly.

For the solution (25), when setting $p = 2$ and considering the field $w_1 = -v_1 - 1$, namely

$$w_1 = -2\chi_x\varphi_y + 2\chi_x\varphi_y \tan h^2(\chi + \varphi + \tau) = -2\frac{4\chi_x\varphi_y \exp[2(\chi + \varphi + \tau)]}{[1 + \exp(2(\chi + \varphi + \tau))]^2}, \tag{28}$$

from which, we can see that choosing $p(x,t) = \exp[2(\chi(x) + \tau_1(t))]$, $q(y,t) = \exp[2(\varphi(y) + \tau_2(t))]$, $\tau(t) = \tau_1(t) + \tau_2(t)$, $a_1 = a_2 = 0$, $a_0 = a_3 = 1$. Thus

$$w_1 = U. \tag{29}$$

Similarly, for the solution (27), if setting $p = 2$ and considering the field $w_2 = -v_2 - 1$, namely

$$w_2 = -2\chi_x\varphi_y + 2\chi_x\varphi_y \cot h^2(\chi + \varphi + \tau) = -2\frac{4\chi_x\varphi_y \exp[2(\chi + \varphi + \tau)]}{[1 + \exp(2(\chi + \varphi + \tau))]^2}, \tag{30}$$

and then choosing $p(x,t) = \exp[2(\chi(x) + \tau_1(t))]$, $q(y,t) = \exp[2(\varphi(y) + \tau_2(t))]$, $\tau(t) = \tau_1(t) + \tau_2(t)$, $a_1 = a_2 = 0$, $a_0 = 1$, $a_3 = -1$, we can see

$$w_2 = U. \tag{31}$$

Therefore, all the localized excitations based on the common formula (2) can be obtained from the solutions (25) and (27). Meanwhile, if selecting the functions $p(x,t)$ and $q(y,t)$ appropriately in the common formula (2), we also can get the nonpropagating solitary wave excitations as follows.

3. (2 + 1)-DIMENSIONAL NONPROPAGATING SOLITARY WAVE EXCITATION

Because of the arbitrariness of the functions $\chi(x)$, $\varphi(y)$ and $\tau(t)$ included in Eqs. (24)–(27), the physical quantities u and v possess quite abundant structures. For example, when $\chi(x) = kx$, $\varphi(y) = cy$, $\tau(t) = dt$, all solutions (24)–(27) become simple travelling wave excitations, which have been obtained in Fan (2003). Here, we are interested in revealing new kinds of (2 + 1)-dimensional nonpropagating solitary wave excitations, such as nonpropagating dromion excitation, nonpropagating compacton excitation, nonpropagating ring excitation, nonpropagating lump excitation, and nonpropagating foldon excitation. For simplification,

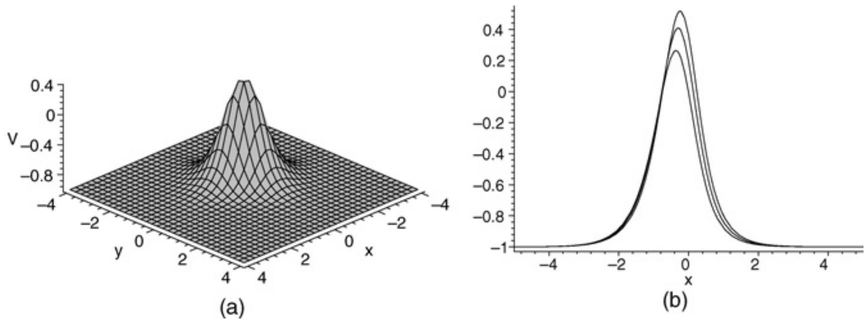

Fig. 1. (a) A nonpropagating dromion excitation plot of the solution v_1 expressed by (32) with condition (33) at time $t = 0$; (b) the corresponding evolutional plot related to (a) at $y = 0$ and at time $t = -3, t = 0, t = 3$.

in the following discussion, we merely analyze the special localized excitations of solution (25), and set $p = 2$, namely

$$V \equiv v_1 = -1 + 2\chi_x\varphi_y - 2\chi_x\varphi_y \tan h^2(\chi + \varphi + \tau). \qquad (32)$$

3.1. Nonpropagating Dromion Excitation

At first, we study the nonpropagating dromion structure. If the functions χ, φ and τ are simply chosen to be

$$\chi = \tan h(x), \quad \varphi = \tan h(y), \quad \tau = \exp(\sin(t)), \qquad (33)$$

then the solution as shown in (32) becomes nonpropagating dromion excitation, which decays exponentially in all directions, depicted in Fig. 1. From the evaluation

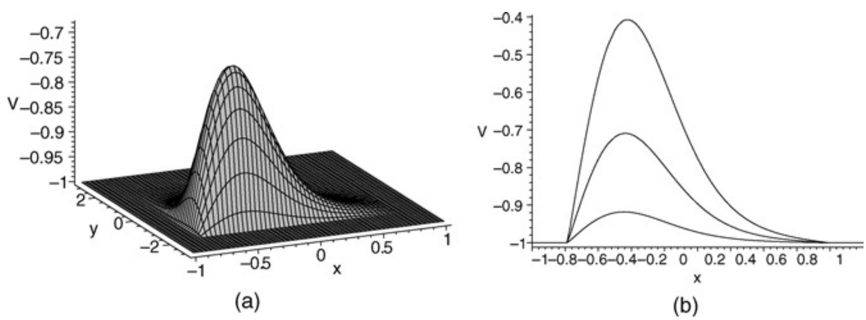

Fig. 2. (a) A nonpropagating compacton excitation plot of the solution v_1 expressed by (32) with condition (34) at time $t = 0$; (b) the corresponding evolutional plot related to (a) at $y = 0$ and at time $t = -100, t = 0, t = 100$.

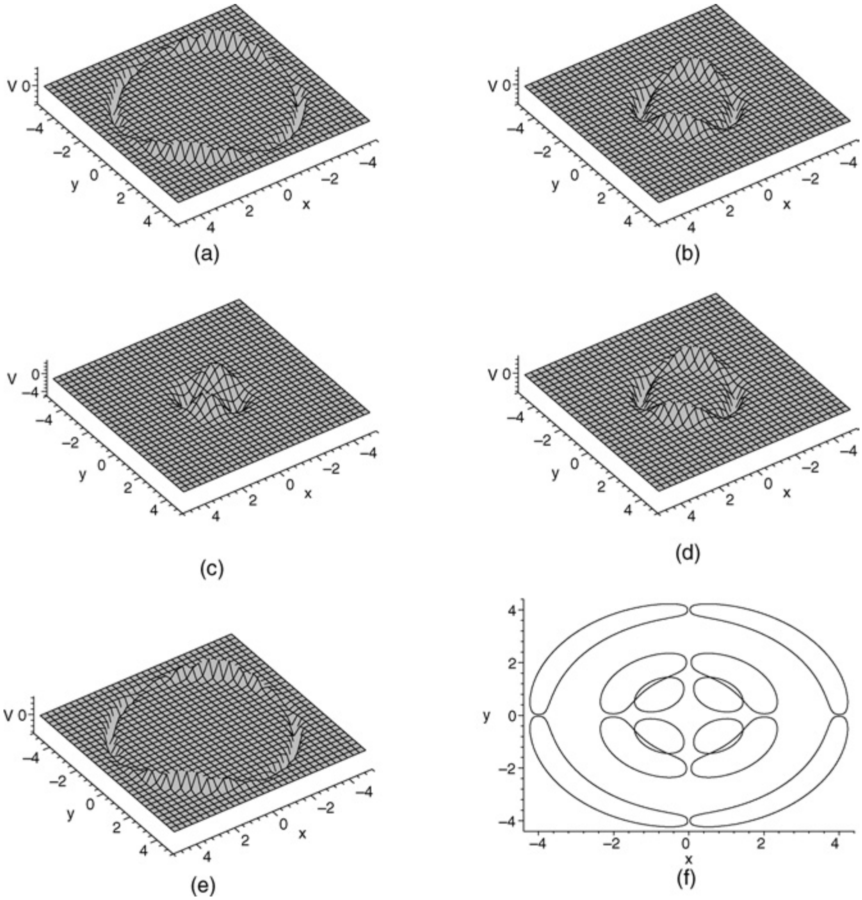

Fig. 3. (a) The nonpropagating ring breather evolutional plots of the solution v_1 expressed by (32) with condition (35) at time $(a)t = -4$, $(b)t = -2$, $(c)t = 0.9$, $(d)t = 2$, $(e)t = 4$, respectively; (b) the contour plot related to (a)–(e) and their values of the contour figure are set $V = -2, 0$.

of Fig. 1(b), we can find that the spatial position of dromion does not change with time, but the amplitude changes with time.

3.2. Nonpropagating Compacton Excitation

If setting

$$\tau = \exp(\sin(t))$$

$$\chi = \begin{cases} 0 & x \le -\frac{\pi}{4} \\ \sin(2x) + 1 & -\frac{\pi}{4} < x \le \frac{\pi}{4} \\ 2 & x > \frac{\pi}{4} \end{cases}$$

$$\varphi = \begin{cases} 0 & y \le -\frac{5\pi}{8} \\ 1.5\sin(0.8y) + 1 & -\frac{5\pi}{8} < y \le \frac{5\pi}{8} \\ 3 & y > \frac{5\pi}{8} \end{cases} \qquad (34)$$

then, we may obtain some types of nonpropagating compacton excitations. The corresponding plot is presented in Fig. 2. From the evaluation Fig. 2(b), we can find that the spatial position of compactons does not change with time, but the amplitude changes with time.

3.3. Nonpropagating Ring Soliton and Standing Breather Excitation

Based on the solution v_1 (32), we can derive the ring soliton excitations. If selecting

$$\chi = -x^2, \quad \varphi = -y^2, \quad \tau = t^2, \qquad (35)$$

then, the solution as shown in (32) becomes nonpropagating ring soliton excitation, which is not equal to zero identically at some closed curves and decays exponentially away from the closed curves. The corresponding evolutional plot is shown in Fig. 3. From Figs. 3(a)–(e), we can find that the spatial position of ring soliton does not change with time. Meanwhile, their contour plot is presented in Fig. 3(f).

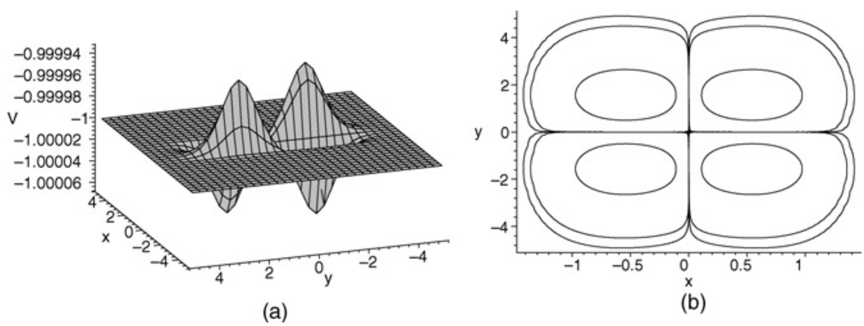

Fig. 4. (a) The nonpropagating lump excitation plot of the solution v_1 expressed by (32) with condition (36) at time $t = 2$; (b) the contour plot related to (a) at time $t = -2, -1, 0, 1, 2$, and its values of the contour figure are set $V = -1.00002, -0.99998$.

3.4. Nonpropagating Lump Excitation

Similarly, we can construct the nonpropagating lump excitation. For example, when $\chi(x)$, $\varphi(y)$ and $\tau(t)$ are chosen

$$\chi = \exp(x^2), \quad \varphi = \exp(-0.5y^2), \quad \tau = t^2, \tag{36}$$

solution (32) becomes nonpropagating lump excitation, which decays algebraically in all directions, as depicted in Fig. 4. From the contour as shown in Fig. 4(b), we can observe that the central position of the lump excitation does not change with time.

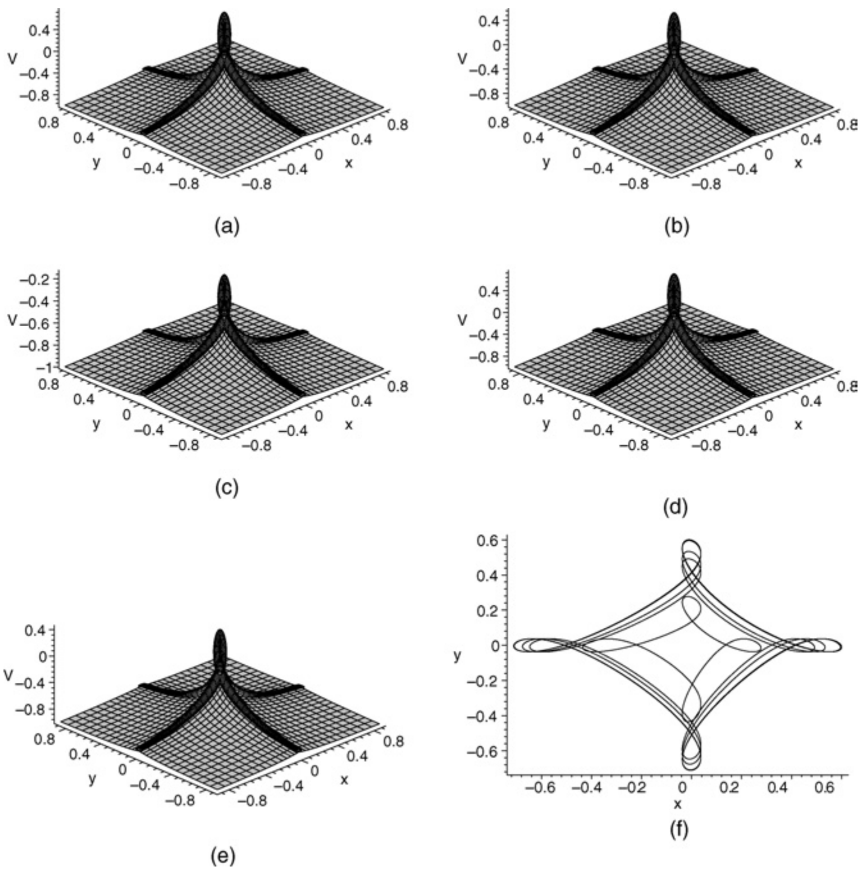

Fig. 5. (a) Evolution plot of a single-foldon solutions determined by (32) with (37) at times (a)$t = -8$, (b)$t = -7$, (c)$t = 0$, (d)$t = 5$, (e)$t = 100$; (b) the contour plot related to (a) at time $t = -8, -7, 0, 5, 100$ and its values of the contour figure are set $V = -0.8$.

3.5. Nonpropagating Foldon Excitation

In order to construct folded excitations, we choose

$$\chi_x = \sec h(\zeta)^2, x = \zeta - 1.15 \tan h(\zeta),$$
$$\varphi_y = \sec h(\eta)^2, y = \eta - 1.15 \tan h(\eta),$$
$$\eta(t) = \exp(\sin(t)), \tag{37}$$

from which, we know that ζ and η are some multi-valued functions in certain regions of x and y, respectively. So the solution v_1 is a multi-valued function of x and y in these regions though it is a single-valued function of ζ and η. The corresponding evolutional plot is presented in Fig. 5. From the Fig. 5(a)–(e), we can find that the central position of the foldon structure does not change with time but the its shape changes with time. Meanwhile, their contour plot is presented in Fig. 5(f).

4. SUMMARY AND DISCUSSION

In summary, with the help of the extended mapping approach, we have successfully realized the variable separation and derived corresponding solutions with three arbitrary functions. By selecting these arbitrary functions properly, a wealth of $(2 + 1)$-dimensional nonpropagating solitons, such as nonpropagating dromion excitation, nonpropagating ring excitation, nonpropagating lump excitation and nonpropagating foldon excitation, are obtained. This method presented in this paper is an initial work, more application to other $(1 + 1)$-dimensional or $(2 + 1)$-dimensional nonlinear physical systems, even higher dimensional nonlinear physical systems should be concerned, and deserve further investigation.

REFERENCES

1. Ablowitz, M. J. and Clarkson, P. A. (1991). *Soliton, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge.
2. Drazin, P. G. and Johnson, R.S. (1989). *Solitons: An Introduction*, Cambridge University Press, Cambridge.
3. Denardo, B. *et al*. (1990). *Physical Review Letters* **64**, 1518.
4. Fan, E. G. (2003). *Journal of Physics A* **36**, 7009.
5. Hong, K. Z., Wu, B., and Chen, X. F. (2003). *Communication in Theoretical Physics* **39**, 393; Ruan, H. Y. and Chen, Y. X. (2001). *Acta Physica Sinca* **50**, 586 (in Chinese); Ruan, H. Y. and Chen,Y. X. (2003). *Chaos, Solitons and Fractals* **17**, 929.
6. Larraza, A. and Putterman, S. (1984). *Journal of Fluid Mechanical* **148**, 443.
7. Lou, S. Y. (1993). *Physical Letter A* **176**, 96.
8. Lou, S. Y. (1994). *Journal of Mathematical Physics* **27**, 3235.
9. Lou, S. Y. (1995). *Mathematical Methods, Applications Science* **18**, 789.
10. Miles, J. W. (1984). *Journal of Fluid Mechanical* **148**, 451.

11. Musette, M. and Conte, R. (1994). *Journal of Physics A: Mathematical General* **27**, 3895; Musette, M., Conte, R., and Pickering, A. (1995). *Journal of Physics A: Mathematical General* **28**, 179; Chen, C. L. and Lou, S.Y. (2003). *Chaos, Solitons and Fractals*. **16**, 27.
12. Paquin, G. and Winternitz, P. (1990). *Physica D* **46**, 122.
13. Peng, Y. Z. (2003). *Communication in Theoretical Physics* **39**, 614.
14. Tang, X. Y., Lou, S. Y., and Zhang, Y. (2002). *Physical Review E* **66**, 046601; Tang, X. Y. and Lou, S. Y. (2003). *Journal Mathematical Physics* **44**, 4000.
15. Tang, X.Y. and Lou, S. Y. (2003). *Communication in Theoretical Physics* **40**, 62.
16. Wu, J. R., Keolian, R., and Rudnich, I. (1984). *Physical Review Letters* **52**, 1421.
17. Wu, J. R. (1985). *Acta Physica Sinica* (in Chinese) **34**, 796; Wang, B. R. *et al*. (1986). *Chinese Physics Letters* **3**, 213; Yan, J. R. and Yang, X. E. (1989). *Chinese Physics Letters* **6**, 537; Chen, W. Z. *et al*., (1995). *Physical Letters A* **208**, 197.
18. Zhang, J. F. (2001). *Chinese Physics* **10**, 0893; Zhang, J. F. (2002). *Chinese Physics* **11**, 0425; Zhang, J. F., Guo, G. P., and Wu, F. M. (2001). *Chinese Physics* **11**, 0533.
19. Zheng, C. L. and Chen, L. Q. (2004). *Journal of the Physical Society of Japan* **73**, 293; Zheng, C. L. and Sheng, Z. M. (2003). *International Journal of Modern Physics B* **17**, 4407.